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Infinite Sets

Introduction

An infinite set is one that is not finite, where a finite set has cardinality of 0 or n $\in \mathbb{N}$. An infinite set can be either countable (for example, the $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$) or uncountable sets (for example $\mathbb{R}$, $\mathbb{C}$, $\mathcal{P}(\mathbb{N})$ or the set of functions from $\mathbb{N}$ to itself). "The use of infinite sets in the definition of the real numbers was one of the reasons behind the renewal of interest in the axiomatic approach to mathematics" [1]. Defining the axioms of the theory of infinite sets is of upmost importance as it can be exploited to give axioms for other theories, such as the theory of sets with exactly n elements, where n is a given positive integer, in a language with equality. Also, the models of this theorem are of more interest and importance than the deductive first-order consequences of the axioms, which are used in other theorems like that of Boolean Algebras or Order Relations.

Axiomatize the theory of infinite sets

The definition of a set requires that there does not exist any duplicates. For a set of cardinality 2, there exists an element x_1 and there exists an element x_2 such that $\neg x_1 = x_2$. This basic definition of a set can be exploited for a set of cardinality n to produce the following sentence:

$$\exists x_1 \exists x_2 ... \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg x_i = x_j \tag{1}$$

This sentence describes a set where there are at least n elements that will be denoted as $\exists_{\geq n}$ henceforth. This sentence can be used to define any set of cardinality n, where $n \in \mathbb{N}$. This definition of a set can be exploited to create a set of axioms Σ where each element of Σ is sentence 1 for every natural number.

$$\Sigma = \{\exists_{\geq n} : n \in \mathbb{N}\} \tag{2}$$

Anything that could model Σ would have a domain with at least n elements for each $n \in \mathbb{N}$, so the domain is infinite as the natural numbers are countably infinite. Likewise any structure with an infinite domain makes each of the sentences in Σ true. Thus Σ axiomatizes the theory of infinite sets [1].

We can axiomatize the theory of sets with exactly n elements, where n is a given positive integer, in a language with equality. Using 1 in conjunction with the statement that the cardinality of the set if equal to n, forcing the 'at least n elements' of Σ to be 'exactly n elements'.

Infinite versus finite axioms

Once it has been shown how to properly axiomatize the theory of infinite sets, we can use the models of the theory to explore the power of the axioms and develop the basis of model theory. One aspect of model theory is the compactness theorem, as stated below. It can be used to test for the existence of an alternative finite set of axioms for the theory of infinite sets.

Compactness theorem

Let Γ be a set of sentences in a first-order language L. Every finite subset of Γ has a model if and only if Γ has a model.

One cannot simply take a finite subset of Σ as any finite subset Δ would have finite model, and thus not axiomatize the theory of infinite sets. This is true as there is a largest n for which $\exists_{\geq n}$ appears in Δ. As any other sentence in Δ is a $\exists_{\geq m}$ for $m < n$, any set with at least this number n of elements would be a model of Δ (from Exercise 6.3 of [1]).

If a finite subset of Γ cannot axiomatize the theory of infinite sets, is it possible that a finite set of sentences axiomatizes the theory? In the finite case, one can string together a finite number of sentences using conjunctions to form a single sentence, σ, that would have the same effect as Σ.

To determine if it is possible to finitely axiomatize the theory of infinite sets, we will work to disprove its contrapositive that there is no finite set of axioms for the theory of infinite sets using previously defined Σ and σ. Consider the set of sentences Γ where Σ and thus σ is an infinite set while $\neg\sigma$ must be something that *is not* an infinite set.

$$\Gamma = \Sigma \cup \{\neg\sigma\} \tag{3}$$

$$= \{\exists_{\geq n} : n \in \mathbb{N}\} \cup \{\neg\sigma\} \tag{4}$$

By Γ's definition it ensures that it has no models, as no model could satisfy the axioms of Σ while also not satisfying the axioms of Σ in the satisfaction of $\neg\sigma$. By the completeness theorem, Γ is inconsistent. Unfortunately, it is possible to show that **any** finite subset of Γ has a model, thus contradicting the Compactness Theorem. This contradiction would show us that the earlier assumption to create σ would be false.

We need to show that **any** finite subset of Γ has a model, so we need to show the worst-case scenario when finite subset Δ includes some sentences of $\exists_{\geq n}$ and the $\neg\sigma$ of Γ. If Δ contains none of $\exists_{\geq n}$s then it is either empty (so vacuously has models) or it just consists of $\neg\sigma$, so any finite non-empty set is a model of Δ. Therefore, Δ has at least this largest n elements as shown previously for a finite subset and it is not a infinite set. An adequate model of Δ would be any n element set. As all n element sets have a model, that means every possible subset Δ of Γ has a model, while Γ has no models. This contradicts the compactness theorem. From all of this we can conclude that we cannot finitely axiomatize the theory of infinite sets.

In conclusion, we have defined infinite sets, axiomatized the theory of infinite sets and proven that this cannot be done with a finite number of axioms.

References

[1] Goldrei D. Propositional and predicate calculus: a model of argument. Springer; 2005.